Generalized tanh Method and Four Families of Soliton-Like Solutions for a Generalized Shallow Water Wave Equation

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We report that a *generalized* tanh method with symbolic computation leads to 4 families of soliton-like solutions for the Jimbo-Miwa's (3 + 1)-dimensional generalized shallow water wave equation.

Key words: Mathematical methods in physics, Generalized tanh method, (3 + 1)-dimensional generalized shallow water wave equation, Symbolic computation, Soliton-like solutions.

During the recent years, aiming at the construction of the travelling-wave or solitary-wave solutions of some nonlinear evolution equations, remarkable attention has been devoted to several versions of the hyperbolic tangent method, or the tanh method, as seen, e.g., in [1–11]. Briefly speaking, the tanh method conjectures *a priori* that a solitary-wave solution $u(x, y, t)$ for a nonlinear evolution equation can be expressed as

$$u(x, y, t) = \sum_{m=0}^{N} a_m \cdot \tanh^m(bx + cy + dt) \qquad (1)$$

and proceeds with the substitution of Ansatz 1 back into the original equation, where N is an integer determined via the balancing act of the highest-order linear and nonlinear contributions, while the a_m's, b, c and d are the constants given by the set of algebraic equations resulting from the equating of the coefficients of $\tanh^m(bx + cy + dt)$'s to zero, with $m = 0, 1, \dots, N$. In addition, an attractive variety of the tanh method [12] has been used as a perturbation technique to derive an approximate solution.

Having investigated the current status of the tanh study, one might ask: "Are the travelling waves the only product of the tanh method? Can we go beyond?"

We next try to answer the above questions by reporting that a generalized tanh method with symbolic computation leads to new soliton-like solutions for a generalized shallow water wave equation, i.e.,

$$u_{yt} + u_{xxxy} - 3\,u_{xx}\,u_y - 3\,u_x\,u_{xy} - u_{xz} = 0\,. \qquad (2)$$

Of current interest in both physics and mathematics are certain higher-dimensional nonlinear evolution discussed, e.g., in [13–17], from which the (1 + 1)-dimensional shallow water wave equations arise as their reductions. In this paper, we study the (3 + 1)-dimensional generalized shallow water wave equation (2) introduced by Jimbo and Miwa [13].

Though originated as the second equation in the Kadomtsev-Petviashvili hierarchy [13], (2) is not completely integrable in the usual sense, as concluded by Dorizzi, Gammaticos, Ramani, and Winternitz [15]. The very recent discussion [17] on the (1 + 1)-dimensional version of (2) arouses the current importance to further investigate (2) itself.

We consider a *generalized tanh* method [18, 19], beginning with the assumption that the soliton-like solutions for certain nonlinear evolution equations, such as (2), being physically localized, are of the x-linear form inside tanh as follows:

$$u(x, y, z, t) = \sum_{m=0}^{N} \mathscr{A}_m(y, z, t) \cdot \tanh^m[\Sigma(y, z, t)\,x + \Theta(y, z, t)]\,, \qquad (3)$$

where $\mathscr{A}_m(y, z, t)$'s, $\Sigma(y, z, t)$ and $\Theta(y, z, t)$ are differentiable functions of y, z and t only, while N is an integer to be determined. Ansatz (3) is obviously more sophisticated than Ansatz (1). The x-linear proposal is

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based on the consideration that in (2), or similar equations, the physical field $u(x, y, z, t)$ has only the first derivative with respect to y, z and t, but higher-order derivatives, with respect to x.

On balancing the highest-order contributions from the linear term (i.e., u_{xxxy}) with the highest order contributions from the nonlinear terms (i.e., $u_x u_{xy}$ and $u_y u_{xx}$), we get $N = 1$, so that Ansatz (3) becomes

$$u(x, y, z, t) = \mathscr{A}(y, z, t) \cdot \tanh[\sigma x + \Theta(y, z, t)] + \mathscr{B}(y, z, t), \quad (4)$$

where $\mathscr{B}(y, z, t) = \mathscr{A}_0(y, z, t)$, $\mathscr{A}(y, z, t) = \mathscr{A}_1(y, z, t) \neq 0$, while $\sigma = \text{constant} \neq 0$ replaces $\Sigma(y, z, t)$ just for simplicity of the future work.

Substituting (4) into (2), using *Mathematicia*, and we get

$$\begin{aligned}
&2\sigma\Theta_z \operatorname{sech}^2(\sigma x + \Theta)\tanh(\sigma x + \Theta) \\
&\quad - \mathscr{A}_z \sigma \operatorname{sech}^2(\sigma x + \Theta) - 2\mathscr{A}_y \sigma^3 \operatorname{sech}^4(\sigma x + \Theta) \\
&\quad + 4\mathscr{A}_y \sigma^3 \operatorname{sech}^2(\sigma x + \Theta)\tanh^2(\sigma x + \Theta) \\
&\quad + 6\mathscr{A} w a_y \sigma^2 \operatorname{sech}^2(\sigma x + \Theta)\tanh^2(\sigma x + \Theta) \\
&\quad - 3\mathscr{A}\mathscr{A}_y \sigma^2 \operatorname{sech}^4(\sigma x + \Theta) \\
&\quad + 2\sigma\Theta_z \operatorname{sech}^2(\sigma x + \Theta)\tanh(\sigma x + \Theta) \\
&\quad + \mathscr{A}_y \Theta_t \operatorname{sech}^2(\sigma x + \Theta) \\
&\quad + 6\mathscr{A}\mathscr{B}_y \sigma^2 \operatorname{sech}^2(\sigma x + \Theta)\tanh(\sigma x + \Theta) \\
&\quad + 16\mathscr{A}\sigma^3\Theta_y \operatorname{sech}^4(\sigma x + \Theta)\tanh(\sigma x + \Theta) \\
&\quad + 12\mathscr{A}^2\sigma^2\Theta_y \operatorname{sech}^4(\sigma x + \Theta)\tanh(\sigma x + \Theta) \\
&\quad - 8\mathscr{A}\sigma^3\Theta_y \operatorname{sech}^2(\sigma x + \Theta)\tanh^3(\sigma x + \Theta) \\
&\quad + \mathscr{A}_t\Theta_y \operatorname{sech}^2(\sigma x + \Theta) \\
&\quad - 2\mathscr{A}\Theta_t\Theta_y \operatorname{sech}^2(\sigma x + \Theta)\tanh(\sigma x + \Theta) \\
&\quad + \mathscr{A}_{yt}\tanh(\sigma x + \Theta) + \mathscr{B}_{yt} \\
&\quad + \mathscr{A}\Theta_{yt}\operatorname{sech}^2(\sigma x + \Theta) = 0. \quad (5)
\end{aligned}$$

Equation to zero the coefficients of like powers of $\tanh(\sigma x + \Theta)$ yields a set of equations, a couple of which comes first, after algebraic manipulations, as

$$\tanh^5(\sigma x + \Theta): \quad (2\sigma + \mathscr{A})\Theta_y = 0, \quad (6)$$

$$\tanh^4(\sigma x + \Theta): \quad (2\sigma + 3\mathscr{A})\mathscr{A}_y = 0. \quad (7)$$

To concurrently satisfy both of them, there are a couple of possibilities

- $\Theta_y = 0$ and $\mathscr{A}_y = 0$,
- $\mathscr{A} = -2\sigma = \text{constant}$,

the second of which then splits into more possibilities. The analysis in the rest of the paper shows that four families of exact solutions result from those possibilities.

Family I: $\Theta_y = 0$ and $\mathscr{A}_y = 0$.

In this place,

$$\tanh^2(\sigma x + \Theta): \quad \mathscr{A}_z = 0 \to \mathscr{A} = \mathscr{A}(t) \text{ only}, \quad (8)$$

$$\tanh^0(\sigma x + \Theta): \quad \mathscr{B}_{yt} = 0 \to \mathscr{B}(y, z, t) = \alpha(y, z) + \beta(z, t), \quad (9)$$

where $\alpha(y, z)$ and $\beta(z, t)$ are the differentiable functions to be determined. Then

$$(10)$$

$$\tanh^3(\sigma x + \Theta) \text{ and } \tanh(\sigma x + \Theta): \quad 3\sigma\alpha_y + \Theta_z = 0,$$

which implies that

$$\alpha_y(y, z) = -\frac{\Theta_z(z, t)}{3\sigma} = \lambda_z(z), \quad (11)$$

a differentiable function of z only.

A couple of the integrations of (11) with respect to y and z respectively yields

$$\alpha(y, z) = \lambda_z(z)\, y + \gamma(z), \quad (12)$$

$$\Theta(z, t) = -3\sigma\lambda(z) + \mu(t). \quad (13)$$

Hence we end up with the first family

$$u^{(I)}(x, y, z, t) = \mathscr{A}(t) \cdot \tanh[\sigma x - 3\sigma\lambda(z) + \mu(t)] + \lambda_z(z)\, y + \kappa(z, t), \quad (14)$$

where the constant σ and the differentiable functions $\mathscr{A}(t)$, $\lambda(z)$, $\mu(t)$, $\kappa(z, t) = \gamma(z) + \beta(z, t)$ all remain arbitrary.

The Case of $\mathscr{A} = -2\sigma = \text{constant}$.

Calculations give rise to

$$\tanh^2(\sigma x + \Theta): \quad \Theta_{yt} = 0 \to \Theta(y, z, t) = \alpha(z, t) + \beta(y, z), \quad (15)$$

$$\tanh^0(\sigma x + \Theta): \quad \mathscr{B}_{yt} = 0 \to \mathscr{B}(y, z, t) = \gamma(z, t) + \mu(y, z), \quad (16)$$

where $\alpha(z, t)$, $\beta(y, z)$, $\gamma(z, t)$ and $\mu(y, z)$ are the differentiable functions to be determined. Then

$$\tanh^3(\sigma x + \Theta) \text{ and } \tanh(\sigma x + \Theta): \quad \sigma\beta_z + \sigma\alpha_z - 4\sigma^3\beta_y - \alpha_t\beta_y + 3\sigma^2\mu_y = 0. \quad (17)$$

Three families of exact solutions will come out from (17).

Family II: $\mathscr{A} = -2\sigma$ and $\beta_y = 0$.

Equation (17) reduces to

$$\alpha_z = -\beta_z - 3\sigma\mu_y = \lambda_z(z), \tag{18}$$

a differentiable function of z only, since the left-hand side is only a function of z, t but the right-hand side is only a function of y, z. Integrating (18) yields

$$\alpha(z, t) = \lambda(z) + \phi(t), \tag{19}$$

$$\mu(y, z) = -\frac{y[\beta_z(z) + \lambda_z(z)]}{3\sigma} + \omega(z). \tag{20}$$

Correspondingly,

$$u^{(II)}(x, y, z, t) = -2\sigma \cdot \tanh[\sigma x + \chi(z) + \phi(t)] - \frac{y\chi_z(z)}{3\sigma} + \kappa(z, t), \tag{21}$$

where the constant σ as well as the differentiable functions $\phi(t)$,

$$\kappa(z, t) = \gamma(z, t) + \omega(z), \quad \text{and} \quad \chi(z) = \beta(z) + \lambda(z), \tag{22}$$

are all arbitrary.

The Case of $\beta_y \neq 0$ with $\mathscr{A} = -2\sigma$

Equation (17) implies that

$$\alpha_t\beta_y - \sigma\alpha_z = \sigma\beta_z - 4\sigma^3\beta_y + 3\sigma^2\mu_y = \lambda_y(y, z), \tag{23}$$

where $\lambda(y, z)$ is a differentiable function of y and z only. Integrating (23) over y yields

$$\lambda(y, z) = \alpha_t(z, t)\beta(y, z) - \sigma\alpha_z(z, t)y - \kappa(z, t), \tag{24}$$

$$\mu(y, z) = \frac{\lambda(y, z)}{3\sigma^2} + \frac{4\sigma}{3}\beta(y, z) - \frac{1}{3\sigma}\int\beta_z(y, z)\,\mathrm{d}y + \phi(z), \tag{25}$$

where $\kappa(z, t)$ and $\phi(z)$ are a couple of differentiable functions.

In addition, the left side of (23) does include certain functions of t also. Its first derivative with respect to t results in another constraint,

$$\alpha_{tt}(z, t)\beta_y(y, z) - \sigma\alpha_{zt}(z, t) = 0, \tag{26}$$

from which the remaining two families appear.

Family III: $\mathscr{A} = -2\sigma$, $\beta_y \neq 0$ but $\alpha_{tt} = 0$.

Corresponding to $\alpha_{tt} = 0$, one has also $\alpha_{zt} = 0$, from (26). Thus,

$$\alpha(z, t) = \psi(z) + \zeta t + \eta, \tag{27}$$

so as to make the third family,

$$u^{(III)}(x, y, z, t) = -2\sigma \cdot \tanh[\sigma x + \zeta t + \beta(y, z) + \psi(z) + \eta] + \frac{4\sigma}{3}\beta(y, z) + \Gamma(z, t) - \frac{1}{3\sigma}\int\beta_z(y, z)\,\mathrm{d}y + \frac{1}{3\sigma^2}[\zeta\beta(y, z) - \sigma y\psi_z(z)], \tag{28}$$

where the constants σ, η, ζ as well as the differentiable functions $\psi(z)$, $\beta(y, z)$ and

$$\Gamma(z, t) = \gamma(z, t) + \phi(z) - \kappa(z, t) \tag{29}$$

are all arbitrary.

Family IV: $\mathscr{A} = -2\sigma$, $\beta_y \neq 0$ and $\alpha_{tt} \neq 0$.

The separation of the variables in (26) comes out with

$$\beta_y(y, z) = \sigma\frac{\alpha_{zt}(z, t)}{\alpha_{tt}(z, t)} = \omega(z) \neq 0, \tag{30}$$

or equivalently,

$$\beta(y, z) = \omega(z)y + \eta(z), \tag{31}$$

$$\sigma\alpha_{zt}(z, t) = \omega(z)\alpha_{tt}(z, t), \tag{32}$$

where $\omega(z)$ and $\eta(z)$ are differentiable functions.

With the introduction of an auxillary variable $v(z, t) = \alpha_t(z, t)$, (32) becomes a first-order linear partial differential equation

$$\sigma v_z(z, t) = \omega(z)v_t(z, t). \tag{33}$$

Solving for it, we get

$$v(z, t) = \frac{\xi}{\sigma}\int\omega(z)\,\mathrm{d}z + \xi t + \tau \tag{34}$$

or

$$\alpha(z, t) = \frac{\xi}{\sigma}t\int\omega(z)\,\mathrm{d}z + \frac{\xi}{2}t^2 + \tau t + \chi, \tag{35}$$

where ξ, χ and τ are constants, with $\xi \neq 0$ since $\alpha_{tt} \neq 0$.

Putting everything together,

$$u^{(IV)}(x,y,z,t) = -2\sigma\cdot\tanh\left[\sigma x + \frac{\xi}{\sigma}t\int\omega(z)\,dz + \frac{\xi}{2}t^2 + \tau t + \omega(z)y + \eta(z) + \chi\right] + \Gamma(z,t) + \frac{4\sigma}{3}\omega(z)y + \frac{\tau\omega(z)y}{3\sigma^2} + \frac{\xi}{3\sigma^3}\omega(z)y\int\omega(z)\,dz - \frac{\omega_z(z)y^2}{6\sigma} - \frac{\eta_z(z)y}{3\sigma}, \quad (36)$$

where the constants $\chi, \tau, \xi \neq 0$, $\sigma \neq 0$ as well as the differentiable functions $\eta(z)$, $\omega(z) \neq 0$,

$$\Gamma(z,t) = \gamma(z,t) + \frac{\xi\eta(z)}{3\sigma^3}\int\omega(z)\,dz + \frac{\xi t\eta(z)}{3\sigma^2} + \frac{\tau\eta(z)}{3\sigma^2} + \frac{4\sigma}{3}\eta(z) - \frac{\kappa(z,t)}{3\sigma^2} + \phi(z) \quad (37)$$

all remain arbitrary.

To sum up, with symbolic computation, our *generalized tanh* method hereby leads to 4 families of new analytic solutions for (2), the Jimbo-Miwa's (3 + 1)-dimensional generalized shallow water wave equation.

Those families, represented by (14), (21), (28) and (36), are potentially interesting in physics since they are soliton-like solutions, although they involve certain arbitrary functions and so are not necessarily bounded everytime.

All the results have been verified using *Mathematica* with respect to (2).

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